Isometric Embeddings between Space Forms

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ABSTRACT

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In 1901 D. Hilbert proved that there is no global isometric immersion of the hyperbolic plane into 3-dimensional Euclidean space, despite the fact that there is a local isometric embedding. Today it is known that there is a global isometric immersion of H^2 into E^5 and a global isometric embedding into E^6, but it is an open problem whether these codimensions can be reduced. More generally, one can ask what the minimal codimensions are for a local embedding, complete immersion and global embedding of a simply connected space form Q_c^n into another $Q_{\tilde{c}}^{n+k}$, where the curvatures c and $\tilde{c}$ are not equal. This thesis seeks to present the best results currently available with regard to this question. In particular, we will show that the problem is solved for all cases apart from global immersions and embeddings of negative extrinsic curvature ($c < \tilde{c}$), when $c \neq 0$. For these unresolved cases we show that the upper bounds for the minimal codimensions are quite small by constructing explicit embeddings and immersions. The worst case is that of embedding H_{-1}^n into $H_{-c^2}^{n+k}$, for $-1 < -c^2 < 0$, where we only know that $n-1 \leq k \leq 5n-5$, and the best unresolved case is that of immersing a sphere S_R^n into a smaller sphere S_r^{n+k}, where we know that the minimal codimension is either n or $n+1$. We also present those non-existence results which are currently known.

Contents

Chapter 1

Introduction

A *space form* is a complete connected Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature c. Up to isometry, there is a unique simply connected space form of dimension n and sectional curvature c, denoted here as Q^n_c. The special cases where c is -1, 0 or 1 are the standard hyperbolic, Euclidean and spherical spaces, denoted H^n, E^n and S^n respectively.

In this thesis we will address the question: when is it possible to isometrically embed, immerse or locally embed Q^n_c into $Q^{n+k}_{\tilde{c}}$, where $c \neq \tilde{c}$. In particular, we would like to know for each case what is the smallest codimension k for which this can be done. Naturally, this codimension must be at least one, because the two manifolds have different sectional curvature. The class of differentiability we are considering is always C^∞. One can obtain isometric embeddings in very low codimensions of class C^1 [14], but they do not preserve the differential geometric nature of the manifold in question, because curvature comes from the second derivative of the metric. It is worthy of note that some of the global results we will give (specifically the immersions and embeddings of the hyperbolic space with negative intrinsic curvature) are not real analytic.

If M^n_c is a space form immersed into another $M^{n+k}_{\tilde{c}}$, then the difference in sectional curvatures, $c - \tilde{c}$, will be called the *extrinsic curvature*. It turns out that the problem of embeddings with positive extrinsic curvature is quite easy, and there is always a global embedding in codimension one. Therefore, the majority of this thesis is a discussion of the more problematic case of negative extrinsic curvature, where, in most cases, only the local problem has really been given a sharp answer. An example of the kind of thing that can happen is the following: the map $f : (-\infty, 0) \times \mathbf{R} \to E^3$ given by

$$f(x, y) = [\int_0^x \sqrt{1 - e^{2t}} dt, e^x e^{iy}],$$

where we have identified E^3 with $\mathbf{R} \times \mathbf{C}$, is an immersion with constant sectional curvature -1. It looks like a horn enclosing the x_1 axis in E^3, with the wide end a circle in the x_2-x_3 plane and the pointed end at $x_1 = -\infty$. It is a part of the hyperbolic plane, but not the whole, because it is not complete, being clearly incomplete as a metric subspace of E^3. It is a famous theorem of Hilbert that there is in fact *no* complete immersion of H^2 into E^3 - even though there is a local embedding. It is also an interesting (and apparently still open) question whether the lowest codimension needed for a complete immersion is the same as that required for a global embedding.

It has been known for some time that there is a C^∞ isometric embedding of any Riemannian manifold into E^n for sufficiently large n [15], [8]. Since, as will be shown below, E^n can be isometrically embedded into H^{n+1}, and also realised as a one-to-one isometric immersion into a sphere of any radius of dimension $4n-1$, this means that any Riemannian manifold can be isometrically embedded into Q_c^{n+k}, for any c, if k is large enough, with the caveat that if the manifold is not compact, and if $c > 0$, then "embedding" must be replaced with "one-to-one immersion". However, these general embedding theorems require a very large codimension, and it turns out that, for specific cases, much better results are obtained by constructing explicit embeddings.

Finally, why is it interesting to isometrically embed only simply connected space forms? The answer is that this is a natural first step to the general problem of embedding space forms, because any space form M_c^n has Q_c^n as its universal cover. Thus, for example, an isometric embedding of Q_c^n into $Q_{\tilde{c}}^{n+k}$ automatically gives an isometric immersion of Q_c^n into any space form $M_{\tilde{c}}^{n+k}$. On the other hand, in order to have a chance of embedding M_c^n into $Q_{\tilde{c}}^{n+k}$, it is necessary that there be an immersion of Q_c^n, which would be the lift of the desired embedding.

1.1 Notation and Preliminaries

M_c^n always stands for a Riemannian manifold of dimension n and constant sectional curvature c. In addition to the designations mentioned above, it is also common to use, for the sphere of radius r, the notation S_r^n instead of $Q_{\frac{1}{r^2}}^n$, and when $c < 0$,

$H^n_c = Q^n_c$. A metric is usually denoted by g and the second fundamental form by h. When a manifold is immersed in another manifold with the induced metric we will usually use g for the metric on both spaces. It is often convenient to use the notation $e^{i\alpha}$ for the vector $[\cos\alpha, \sin\alpha]$.

We will generally assume those definitions and results commonly taught in first year graduate courses on Geometry and Topology. Assumptions more specific to Riemannian Geometry can be found, for example, in [7]. The most important of these are the definition of the curvature tensor R, which to every X and Y in the tangent space T_pM assigns a mapping $R(X,Y) : T_pM \to T_pM$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

(where the expression does not depend on the extension of X, Y and Z to vector fields in a neighbourhood of p), and Gauss' equation

$$g(R(X,Y)Z,T) = g(\tilde{R}(X,Y)Z,T) + g(h(X,T),h(Y,Z)) - g(h(Y,T),h(X,Z)),$$

which relates, through the second fundamental form h, the curvature tensor $\tilde{R}$ of a manifold $\tilde{M}$ to the curvature tensor R of a submanifold M. The normal components of $\tilde{R}$ will satisfy the Codazzi equation:

$$(\tilde{R}(X,Y)Z)^\perp = (\nabla^\perp_X h)(Y,Z) - (\nabla^\perp_Y h)(X,Z),$$

$$(\nabla^\perp_X h)(Y,Z) := \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Chapter 2

Positive Extrinsic Curvature - Umbilic Hypersurfaces

In this chapter we will show that if $\tilde{c} < c$ then there is always a codimension 1 global isometric embedding of Q_c^n into $Q_{\tilde{c}}^{n+1}$. Note that in this case the second fundamental form is real-valued and, choosing an orthonormal basis in which it is diagonal, Gauss' equation is

$$c - \tilde{c} = h(e_i, e_i)h(e_j, e_j).$$

Since this must hold for all $i \neq j$, it follows (at least for $n > 2$) that $h(e_i, e_i) = \sqrt{c - \tilde{c}}$ for all i, and that the embedding will necessarily be *totally umbilic*. An isometric immersion M^n into $\tilde{M}^{n+1}$ is totally umbilic if at every point p of M, the second fundamental form is proportional to the metric: $\tilde{g}(h(X,Y),\eta)(p) = \lambda g(X,Y)(p)$, where η is a unit normal and $\lambda \in \mathbf{R}$. When the ambient space has constant sectional curvature, it follows that λ is a constant, independent of p ([7], page182).

2.1 $\mathbf{0 < \tilde{c} < c}$: Spheres inside Larger Spheres

In this case we want to embed a sphere into a higher dimensional sphere of lower sectional curvature. This can be done as follows: $Q_{\tilde{c}}^{n+1}$ is the sphere of radius $\sqrt{\frac{1}{\tilde{c}}}$, given by $\{x \in \mathbf{R}^{n+2} \mid x_1^2 + ... + x_{n+2}^2 = \frac{1}{\tilde{c}}\}$. For any $c > \tilde{c}$, the set

$$S = \{(x_1, x_2, ..., x_{n+1}, \sqrt{\tilde{c}^{-1} - c^{-1}}) \in \mathbf{R}^{n+2} \mid x_1^2 + ... + x_{n+1}^2 = \frac{1}{c}\}$$

is a sphere of radius $\sqrt{\frac{1}{c}}$ in $\mathbf{R}^{n+1} \subset \mathbf{R}^{n+2}$ centred at $(0, ..., 0, \sqrt{\tilde{c}^{-1} - c^{-1}})$. Since $x \in S$ satisfies $x_1^2 + ... + x_{n+2}^2 = \frac{1}{\tilde{c}}$, S is a copy of Q_c^n sitting inside $Q_{\tilde{c}}^{n+1}$. Both spheres have metrics induced from that of E^{n+2}, so the embedding is isometric.

2.2 $\mathbf{\tilde{c} = 0 < c}$: Spheres into Euclidean Space

Q_c^n is the standard sphere of radius $\sqrt{\frac{1}{c}}$ centred at the origin, $S^n_{\sqrt{\frac{1}{c}}} \subset E^{n+1}$.

2.3 $\mathbf{\tilde{c} < 0,\ \tilde{c} < c}$: Hypersurfaces of $\mathbf{H^n}$

We will show that one can isometrically embed a complete simply connected space form of arbitrary curvature $c > -1$ as a totally umbilic hypersurface of H^{n+1}, the standard hyperbolic space of constant curvature -1. After rescaling the metrics (multiplying the metric by k changes the sectional curvatures by $\frac{1}{k}$), this takes care of all cases.

We take as our definition of H^n the half-space $\{x \in \mathbf{R}^n \mid x_n > 0\}$, with the metric $\bar{g}_{ij}(x_1, ..., x_n) = x_n^{-2} g_{ij}(x_1, ..., x_n)$, where $g_{ij} = \delta_{ij}$, the Euclidean metric. See [7] Chapter 8 for a proof that H^n is complete and has constant sectional curvature -1, as well as for further details of the following useful facts: two metrics $\bar{g}$ and g on a manifold M are said to be *conformal* if $\bar{g} = \mu g$ for some positive function $\mu : M \to \mathbf{R}$. Essential to our construction is the fact that an umbilic immersion is still umbilic after a conformal change of the ambient metric: suppose we have an umbilic immersion so that $\tilde{g}(h(X, Y), \eta) = \lambda \tilde{g}(X, Y)$, where η is a unit normal. If $\bar{\tilde{g}} = \mu \tilde{g}$ is the conformal change, then $\frac{\eta}{\mu}$ is the unit normal with respect to $\bar{\tilde{g}}$ and, after calculating the corresponding second fundamental form we obtain:

$$\bar{\tilde{g}}(\bar{\nabla}_X Y, \frac{\eta}{\mu}) = \bar{\lambda} \bar{\tilde{g}}(X, Y), \tag{2.1}$$

$$\bar{\lambda} = \frac{2\lambda\mu - \eta(\mu)}{2\mu\sqrt{\mu}}. \tag{2.2}$$

Now consider the hypersurface $S^n(t) \subset \mathbf{R}^{n+1}$, a sphere of radius 1 centered at $(0, 0, ..., 0, 1 + t)$. It is a totally umbilic hypersurface of E^{n+1} with $\lambda = 1$. Our metric $\bar{\tilde{g}}$ on H^{n+1} is conformal to the Euclidean metric with $\mu = x_{n+1}^{-2}$. Therefore the intersection of $S^n(t)$ with the upper half-space is also a totally umbilic hypersurface $\bar{S}^n(t)$ of H^{n+1}. If η is a unit normal to this sphere in E^{n+1}, then $\bar{\eta} = x_{n+1}^2 \eta$ is the unit normal in H^{n+1}, and using (2.1) we have the second fundamental form

$$\bar{h}(X, Y) = \bar{\lambda} \bar{\tilde{g}}(X, Y),$$

$$\bar{\lambda} = \frac{2x_{n+1}^{-2} - \eta(x_{n+1}^{-2})}{2x_{n+1}^{-3}}.$$

We know $\bar{\lambda}$ is constant, so we can calculate it conveniently at a point such as $p = (1, 0, ..., 0, 1+t)$ where $\eta(x_{n+1}) = 0$, obtaining

$$\bar{h}(X, Y) = (1+t)\bar{\bar{g}}(X, Y).$$

It is now easy to compute the sectional curvatures of $\bar{S}^n(t)$ using Gauss' equation:

$$\begin{aligned} c &= -1 + (1+t)^2 \\ &= t^2 + 2t. \end{aligned}$$

The map $[t \mapsto t^2 + 2t]$ is a one-to-one increasing bijection from $(-1, \infty) \to (-1, \infty)$, with inverse $t = -1 + \sqrt{1+c}$. We thus have, for every $c > -1$, a totally umbilic hypersurface $S_c = \bar{S}^n(-1 + \sqrt{1+c})$ of H^{n+1}, with constant sectional curvature c.

It is not hard to see that these hypersurfaces are complete: recall first that by a theorem of Hopf and Rinow (see [7]) geodesic completeness is equivalent to metric completeness with respect to the distance function induced by the Riemannian metric. Now suppose that $\{p_n\}$ is a Cauchy sequence in S_c. Because small Euclidean distances correspond to arbitrarily large hyperbolic distances as you move closer to the plane $x_{n+1} = 0$, it is easy to show that the Cauchy property implies that there exists an $\epsilon > 0$ such that the entire Cauchy sequence is contained in the half-space $H_\epsilon := \{x_{n+1} > \epsilon\}$. Now the sphere $S^n(t)$ is complete and so, therefore, is the closed subset $S^n(t) \cap \bar{H}_\epsilon$ (where the closure $\bar{H}_\epsilon = \{x_{n+1} \geq \epsilon\}$). But on $H_{\frac{\epsilon}{2}}$, which contains this set, the metric of S_c is conformal to the spherical metric induced from E^{n+1} by a conformal factor which is both bounded above and bounded below by a positive constant. This means that the two distance functions are equivalent, so that convergence of Cauchy sequences coincides for the two metrics. In other words, $S^n(t) \cap \bar{H}_\epsilon$ is also complete in the metric of S_c, and therefore the sequence converges.

Chapter 3

Negative Extrinsic Curvature: Local Results

From now on, we will only be dealing with the case $c < \tilde{c}$.

3.1 A Non-Existence Result

Proposition 3.1 *(Otsuki [17], Cartan [5].) Let $c < \tilde{c}$. There is no (local) isometric embedding from M_c^n into $\tilde{M}_{\tilde{c}}^{2n-2}$.*

Proof: Let $p \in M$. For any orthonormal vectors $X, Y \in T_pM$, Gauss' equation says:

$$0 > c - \tilde{c} = \tilde{g}(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2. \tag{3.1}$$

The key point in the proof is that given any X in T_pM, the codimension $n-2$ is small enough to guarantee that we can always find some $Y \in T_pM$, linearly independent from X, such that the last term in (3.1) is zero. If we then choose our X to minimize a judiciously chosen function f on the (compact) unit sphere UT_pM in T_pM, then the first term on the right hand side of (3.1) will be positive, a contradiction.

Firstly, we can write $h = h^1\xi_1 + ... + h^{n-2}\xi_{n-2}$, where each h^i is a symmetric bilinear form on T_pM and ξ_i span the normal space at p. We have, for fixed $X \in T_pM$, the vector spaces $V_i = \{Z \in T_pM \mid h^i(X,Z) = 0\}$ of dimension at least $n-1$. Thus the space $\{Z \in T_pM \mid h(X,Z) = 0\}$, being the intersection of these, has dimension at least $n-(n-2) = 2$. Consequently, there is a vector Y, linearly independent from X, such that $h(X,Y) = 0$. Note that, with (3.1), this implies that $h(X,X)$ is non-zero for any $X \in T_pM$.

Now take X to be a minimum for the map $f : UT_pM \to \mathbf{R}$ taking $X \mapsto \tilde{g}(h(X,X), h(X,X))$. The assumption that X is a critical point for this map implies that the vector Y found above is orthogonal to X. To see this, we can differentiate f using the fact that $\tilde{g}$ and h are symmetric and bilinear to get:

$$df_X(Z) = 4\tilde{g}(h(X,X), h(X,Z)).$$

This expression must be zero for any $Z \in T_X(UT_pM) = \{Z \in T_pM \mid g(X,Z) = 0\}$. Now write $Y = aX + Z$, with $g(X,Z) = 0$. Since $h(X,Y) = 0$, we have

$$\begin{aligned} 0 &= \tilde{g}(h(X,X), h(X,Y)) \\ &= a\tilde{g}(h(X,X), h(X,X)) + \tilde{g}(h(X,X), h(X,Z)). \end{aligned}$$

Our choice of Z makes the last term zero, and we previously observed that $h(X,X)$ is non-zero. It follows that a must be 0 and Y is thus orthogonal to X. Normalizing Y to be of unit length, this gives us a curve $\gamma(t) = \cos(t)X + \sin(t)Y$ in UT_pM, with $\gamma(0) = X$.

Finally, we use the fact that f has a minimum on UT_pM at X, and the second derivative test on $f(\gamma(t))$, namely:

$$\frac{d^2}{dt^2}|_{t=0} f(\gamma(t)) \geq 0.$$

To compute this inequality, observe that on expanding $f(\gamma(t))$ half the terms are zero because they contain $h(X,Y)$ and we are left with

$$\begin{aligned} f(\gamma(t)) &= \cos^4(t)\tilde{g}(h(X,X), h(X,X)) + \sin^4(t)\tilde{g}(h(Y,Y), h(Y,Y)) \\ &\quad + \cos^2(t)\sin^2(t)\tilde{g}(h(X,X), h(Y,Y)). \end{aligned}$$

At $t = 0$ the second derivatives of these trigonometric coefficients are -4, 0 and 2 respectively, so the inequality becomes

$$-4\tilde{g}(h(X,X), h(X,X)) + 2\tilde{g}(h(X,X), h(Y,Y)) \geq 0.$$

In other words, $\tilde{g}(h(X,X), h(Y,Y)) > 0$, contradicting (3.1). ♠

Remark: The assumption of constant sectional curvatures is not necessary. The proof holds as long as at one point p the difference between the sectional curvature of M and of $\tilde{M}$ is strictly negative for any 2-plane in T_pM.

3.2 The Second Fundamental Form

In the next section we will prove that the result in the previous section is sharp, that is to say, there is always an embedding of a neighbourhood of any point in M_c^n into $\tilde{M}_{\tilde{c}}^{2n-1}$. Before proving this, we will first study the local structure of the second fundamental form in codimension $n-1$. This will be useful for global non-immersion results later.

Most of the results, both local and global, in this critical codimension rely to a great extent on work of Elie Cartan and, later, John Douglas Moore, which resulted in the following interesting fact:

Theorem 3.1 *(J.D. Moore [13].) Suppose M_c^n is isometrically immersed in $\tilde{M}_{\tilde{c}}^{2n-1}$. Then:*

1. *Both the first and second fundamental forms of the immersion can be diagonalized simultaneously. The frame, $e_1, ..., e_n$, which does this is unique up to permutations and changes of signs.*

2. *Locally there is an orthogonal coordinate system (principal coordinates) whose tangent directions are e_i. If M is simply connected these coordinates are well-defined globally.*

3. *If M is complete and simply connected then the the principal coordinates define a diffeomorphism from R^n to M.*

The proof depends on an algebraic result of E. Cartan, which we will state here, but postpone the proof of until the end of this section. Let V be an n-dimensional real vector space and $\Phi^1,\ \Phi^2,\ ...,\ \Phi^n$ be symmetric bilinear forms on V. Φ^i are said to be *exteriorly orthogonal* if, for all $X,\ Y,\ Z,\ W \in V$ we have

$$\sum_{i=1}^{n}[\Phi^i(X,Y)\Phi^i(Z,W) - \Phi^i(X,W)\Phi^i(Z,Y)] = 0. \tag{3.2}$$

Theorem 3.2 *(E. Cartan [5].)* *Let $\Phi^1, ..., \Phi^n$ be n exteriorly orthogonal symmetric bilinear forms on V, which in addition have the following property: whenever X is a vector such that $\Phi^i(X,Y) = 0$ for all $1 \leq i \leq n$ and for all $Y \in V$, then $X = 0$. Then there exists a real orthogonal matrix A and n linear functionals $\phi^1, ..., \phi^n$ such that*

$$\Phi^i = \sum_j A^i_j \phi^j \otimes \phi^j, \qquad\qquad 1 \leq i \leq n.$$

In other words, Φ^i are simultaneously diagonalized with respect to the basis dual to ϕ^i.

Note that if we have an immersion M^n_c into N^{2n}_c then Gauss' equation applied with an orthonormal frame η_i for the normal bundle exactly says that the n components, $h^i = g(h, \eta_i)$, of the second fundamental form are exteriorly orthogonal on the tangent space, T_pM, at a point p. If, moreover, one of the h^i is definite, the second condition required in the theorem will be satisfied.

We can achieve this situation as follows: locally $\tilde{M}^{2n-1}_{\tilde{c}}$ is simply connected, so we can regard our immersion as $M_c \to Q^{2n-1}_{\tilde{c}} \to Q^{2n}_c$, where the last map is the umbilic inclusion described in chapter 2. Choose a basis η_i for the normal space such that

η_n is the normal to $Q_{\tilde{c}}^{2n-1}$ in Q_c^{2n}. The second fundamental form of the immersion M into Q_c^{2n} is of the form $h = \tilde{h} + h^n \eta_n$, where $\tilde{h}$ is the second fundamental form of the original immersion into $Q_{\tilde{c}}^{2n-1}$, and $h^n \eta_n$ is the second fundamental form of the umbilic inclusion. Then $h^n = \sqrt{\tilde{c} - c}g$, a positive constant multiplied by the metric of Q_c^{2n}, and is therefore positive definite. Applying the theorem, we can now assume that each h^i is diagonal. Since h^n is proportional to the metric, the diagonalizing basis for T_pM also diagonalizes the metric, and, after rescaling, we have an orthonormal basis $e_1,\ \ldots,\ e_n$.

We still need to prove the uniqueness of e_i. Choose a basis for the normal space to makes things simple. Consider the vectors $N_i = h(e_i, e_i) = \sum_j h^j(e_i, e_i)\eta_j$, where η_j are as above. Since its last component is $h^n(e_i, e_i) = \sqrt{\tilde{c} - c}g(e_i, e_i)$, it follows that each N_i is certainly non-zero. Moreover, Gauss' equation says

$$0 = g(h(e_i, e_i), h(e_j, e_j)) - ||h(e_i, e_j)||^2,$$

where the last term is zero because h is diagonal, implying that N_i are mutually orthogonal. Therefore, we can take $\xi_i = N_i/||N_i||$ as an orthonormal basis for the normal space. In this basis,

$$h(e_i, e_j) = \delta_{ij} \frac{\sqrt{\tilde{c} - c}}{\alpha_i} \xi_i,$$

where

$$\alpha_i = \frac{\sqrt{\tilde{c} - c}}{||N_i||}, \qquad \sum_i \alpha_i^2 = 1.$$

The last identity follows from the fact that α_i are in fact the components of the unit umbilical normal η^n with respect to the basis ξ_i, calculated by $g(\eta^n, \xi_i) = g(\eta^n, h(e_i, e_i)/||N_i||) = \sqrt{\tilde{c} - c}/||N_i||$.

Now each vector e_i is the only eigenvector with non-zero eigenvalue for the matrix $h^i = g(h, \xi_i)$. Therefore, up to changes of sign, the basis $\{e_i\}$ is the only basis which simultaneously diagonalizes h^i in this normal frame. But a change of normal frame does not affect whether h^i are diagonal or not, which proves uniqueness. Thus we have proved the first part of Theorem 3.1.

For the second part, it is straightforward to verify that Codazzi's equations are equivalent to

$$\nabla_{e_i} e_j = \sum_k \tau_j^k(e_i) e_k, \tag{3.3}$$

$$\nabla^{\perp}_{e_i} \xi_j = \sum_k \omega_j^k(e_i) \xi_k, \tag{3.4}$$

$$\tau_i^j(e_j) = \frac{1}{\alpha_j} e_i(\alpha_j), \tag{3.5}$$

$$\omega_i^j(e_j) = \frac{1}{\alpha_j} e_j(\alpha_i), \tag{3.6}$$

where $i \neq j$ and the connection coefficients not stated are all zero. Using this, it is easy to check that the orthogonal (but not orthonormal) vector fields

$$X_i = \alpha_i e_i$$

have mutually vanishing Lie brackets. By Frobenius' theorem one can therefore integrate them to get a local orthogonal coordinate system.

Finally, if M is simply connected, a consistent global choice of frame $\{e_i\}$ (and therefore X_i) is arranged as follows: $\{e_i\}$ are uniquely defined at any point, up to permutations and changes of sign. Fix a choice at some point p. If q is any other point, take a path from p to q. On a small neighbourhood of p there is only one choice of smooth frame $\{e_i\}$ which will agree with our choice at p, because permuting or changing signs is not a continuous operation. We can continue this along the path to q, to get a unique choice at q. This is independent of path, because M is simply connected, so any path from p to q can be continuously deformed into any other, taking the frame with it.

The last statement of the theorem follows because if M is complete then the flows $\phi_i(x_0, t_i)$, which are the principle coordinates through some point x_0, are complete. Hence one can show that the map

$$\Phi(t_1, ..., t_n) = \phi_1(...\phi_{n-1}(\phi_n(x_0, t_n), t_{n-1})...t_1)$$

is a covering map $R^n \rightarrow M$. ♠

Remark: The presentation given here differs slightly from Moore's, in that we make use of the umbilic inclusion to get simple expressions for the connection coefficients. This analysis has been known for some time, and can be found in [6].

Proof of Theorem 3.2: We will show that the theorem follows from the fact that a collection of matrices which commute pairwise can be simultaneously diagonalized. The bilinear form Φ^i may be regarded as a linear map $V \rightarrow V^*$, inducing a linear map $\Phi^i \wedge \Phi^i : V \wedge V \to V^* \wedge V^*$, which acts by the rule:

$$\Phi^i(x) \wedge \Phi^i(y)(z \wedge w) = \frac{1}{2}[\Phi^i(x,z)\Phi^i(y,w) - \Phi^i(x,w)\Phi^i(y,z)].$$

Thus, Φ^i are exteriorly orthogonal if and only if

$$\sum_i \Phi^i \wedge \Phi^i = 0.$$

Setting $\mathbf{\Phi} = [\Phi^1, ..., \Phi^n]^t$, we are required to prove that there exists a real orthogonal matrix A such that $\mathbf{\Phi} = A\mathbf{\Psi}$, where each component $\Psi^j = \pm\theta^j \otimes \theta^j$, and θ^j are linear functionals. The last condition is equivalently stated as

$$\Psi^j \wedge \Psi^j = 0. \tag{3.7}$$

Fact: There exists a vector $X \in V$ such that $\Phi^i(X)$, $i = 1, ..., n$, are linearly independent.

We will prove the fact later. Now let $v_1, ..., v_n$ be a basis for V such that $\Phi^i(v_1)$ are linearly independent. Then we can write

$$\Phi^i(v_k) = \sum_j C(k)^i_j \Phi^j(v_1).$$

Substituting this expression and v_1 into the condition (3.7),

$$\sum_i \Phi^i(v_1) \wedge \Phi^i(v_k) = 0,$$

we see that the matrices $C(k)$ are symmetric. Now do the same with v_j and v_k, to get

$$\sum_{i,m,n} C(j)^i_m C(k)^i_n \Phi^m(v_1) \wedge \Phi^n(v_1) = 0,$$

which means that

$$C(j)^i_m C(k)^i_n = C(j)^i_n C(k)^i_m.$$

Together with the fact that $C(i)$ are symmetric matrices, this implies that they all commute pairwise. Therefore they can be simultaneously diagonalized, i.e. there is an orthogonal matrix A, such that $AC(k)A^t = D(k)$ is diagonal for all k. Now set $\mathbf{\Psi} = A\mathbf{\Phi}$, and we have

$$\begin{aligned}\mathbf{\Psi}(v_k) &= AC(k)\mathbf{\Phi}(v_1) \\ &= AC(k)A^t\mathbf{\Psi}(v_1) \\ &= D(k)\mathbf{\Psi}(v_1).\end{aligned}$$

It follows that $\mathbf{\Psi}(v_i) \wedge \mathbf{\Psi}(v_j) = 0$, because each component is of the form $\lambda_1\Psi^i(v_1) \wedge \lambda_2\Psi^i(v_1)$.

It remains to prove the existence of a vector X such that $\Phi^i(X)$ are linearly independent. Assume the contrary. For any vector $Z \in V$, let $U(Z)$ be the subspace of V^* generated by $\Phi^i(Z)$, $i = 1, ..., n$, and let M be a vector such that $U(M)$ has maximal dimension $p < n$. Without loss of generality, assume that $\Phi^1(M), ..., \Phi^p(M)$ are linearly independent, and $\Phi^i(M) = 0$ for $i > p$. For any other vector Y, we have

$$\sum_i^p \Phi^i(M) \wedge \Phi^i(Y) = 0,$$

thus for $i \leq p$, we can write

$$\Phi^i(Y) = \sum_{j=1}^p c^i_j \Phi^j(M),$$

where c^i_j are symmetric. This means that the subspace W of V^* defined by

$$W = \{\Phi^i(Z)|\ Z \in V,\ i \leq p\}$$

is of dimension p, and therefore there is a vector Z in V which is annihilated by W. By the second hypothesis of the theorem, there exists λ and a vector $N \in V$ such that $\Phi^\lambda(Z, N) \neq 0$, and it follows that $p < \lambda \leq n$. If ϵ is sufficiently small, the covectors $\Phi^i(M + \epsilon N) = \Phi^i(M) + \epsilon\Phi^i(N)$, $1 \leq i \leq p$, will still be linearly independent, and so they will generate the p-dimensional subspace W, since they all lie inside W. But $\Phi^\lambda(M + \epsilon N) = \epsilon\Phi^\lambda(N)$ is non-zero and not in W, and so $U(M + \epsilon N)$ has dimension at least $p + 1$, a contradiction. ♠

3.3 Explicit Local Embeddings

There is more than one way to prove that local embeddings of negative extrinsic curvature always exist in codimension $n-1$. One possible proof would proceed as follows: using the coordinates and connection coefficients described in the previous section, one can show that the Gauss, Codazzi and Ricci equations for such an embedding are equivalent to a system of non-linear PDEs for the functions α_i and some functions f_{ij} which are derivatives of α_i in the principle directions. One can then show that this so-called "generalized sine-Gordon equation" does indeed have a local solution, given arbitrary initial conditions along certain curves [6]. The solutions then integrate up, using the fundamental theorem of submanifold theory, to give a local embedding with the corresponding extrinsic curvature.

Rather than go through the details of that here, we will prove the same fact by giving explicit local embeddings of Q_c^n into $Q_{\tilde{c}}^{2n-1}$ for $c < \tilde{c}$.

3.3.1 $0 < c < \tilde{c}$: A Portion of a Sphere Embedded into a Smaller Sphere

It will be convenient here to use the warped product representation of the unit sphere S^n, $n > 1$, namely the set $(0, \pi) \times S^{n-1}$, with the metric $dx^2 + \sin^2(x)d\sigma^2$, where $d\sigma^2$ is the metric on S^{n-1}. See, for example, [18], page 14, for an embedding of this metric into S^n. We are going to isometrically embed a small open subset of S^n into S_R^{2n-1}, where R is an arbitrary number between 0 and 1. After rescaling, this proves all the cases.

Proceeding by induction on n, it is clear that a sufficiently small portion of S^1 can be embedded isometrically into S_R^1 for any $R < 1$. So assume that we have an open set $U \subset S^{n-1}$ and, for some arbitrarily small positive r, an embedding $f : U \to S_r^{2n-3} \subset E^{2n-2}$, such that the pullback metric is $f^* g_{E^{2n-2}} = d\sigma^2$ (the metric of S^{n-1}). Now define $F : (0, \pi) \times U \to E^{2n}$ by

$$F(x, \mathbf{y}) = [\alpha(x)e^{i\beta(x)}, \sin(x)f(\mathbf{y})],$$

where the functions α and β are to be defined. We have

$$\begin{aligned} g(F, F) &= \alpha^2(x) + \sin^2(x) g(f(y), f(y)) \\ &= \alpha^2(x) + \sin^2(x) r^2, \end{aligned}$$

which follows from our assumption on f. We want $f(U) \subset S_R^{2n-1}$, so it is required that the right hand side of the above equation be equal to R^2. Solving this for α, we obtain

$$\alpha(x) = \sqrt{R^2 - r^2 \sin^2(x)}.$$

This is a well defined analytic function for sufficiently small r. We also want the pull-back metric to be $dx^2 + \sin^2(x) d\sigma^2$. Writing $d\sigma^2 = \sum_{i,j} \sigma_{ij} dy^i dy^j$, for some coordinates y_j on U, we immediately have

$$\begin{aligned} g(F_{y_i}, F_{y_j}) &= \sin^2(x) \sigma_{ij}, \\ g(F_x, F_{y_i}) &= \cos(x) \sin(x) g(f, f_{y_i}) \\ &= 0. \end{aligned}$$

The last equality is due to the fact that f takes its values in a sphere, and f_{y_i} is tangent to the sphere. Thus, all that is required now is the following equality:

$$g(F_x, F_x) = (\alpha'(x))^2 + (\alpha(x)\beta'(x))^2 + r^2 \cos^2(x) = 1.$$

Solving this for β' we require

$$\beta'(x) = \sqrt{\frac{1 - (\alpha'(x))^2 - r^2 \cos^2(x)}{(\alpha(x))^2}}.$$

Now $\alpha'(x) = \frac{-r^2 \sin(2x)}{2\alpha(x)}$, so the equation above becomes

$$\beta'(x) = \frac{1}{\alpha(x)} \sqrt{1 - r^2(\cos^2 x + \frac{r^2 \sin^2(2x)}{4\alpha^2(x)})}.$$

Observing that $\alpha(x)$ is arbitrarily close to R in value, for sufficiently small r, it follows that $\beta'(x)$ is well defined for all $x \in (0, \pi)$, and so is $\beta(x) = \int_0^x \beta'(t)dt$ on the same interval. In other words, this is an isometric embedding $(0, \pi) \times U \to S_R^{2n-1}$.

3.3.2 $\mathbf{c = 0 < \tilde{c}}$: The Clifford Torus

Consider the Clifford torus $f: \ E^n \to S_R^{2n-1} \subset E^{2n}$, given by

$$f(x_1, ..., x_n) = \frac{R}{\sqrt{n}}(e^{i\frac{\sqrt{n}}{R}x_1}, ..., e^{i\frac{\sqrt{n}}{R}x_n}).$$

This satisfies $g(f, f) = R^2$, so f is a map from E^n into S_R^{2n-1}, and $g(f_{x_i}, f_{x_j}) = \delta_{ij}$, so it is an isometry. f is periodic in each x_i, with period $P = \frac{2\pi R}{\sqrt{n}}$. When restricted to a fundamental domain $D = (0, P) \times .... \times (0, P)$ it is an embedding.

3.3.3 $\mathbf{c < 0}$: Local Embeddings of Hyperbolic Space

For these embeddings we will use the complete hyperbolic metric on $\mathbf{R}^n$ given by $dx^2 + e^{2x}(dy_1^2 + ... + dy_{n-1}^2)$. The map $\phi : (\mathbf{R}^n, g) \to H^n, \ \ \phi(x) = (e^{-x_1}, x_2, ..., x_n)$ is easily checked to be an isometry with the upper half space model $H^n = \{x \in \mathbf{R}^n \mid x_1 > 0\}$, with the metric $g_{H^n} = x_1^{-2}(dx_1^2 + \ ... \ + dx_n^2)$.

There is a well-known (non-complete) embedding of our metric into E^{2n-1}, defined by $f : (-\infty, 0] \times \mathbf{R}^{n-1} \to E^{2n-1}$,

$$f(x, y_1, ..., y_{n-1}) = [\int_0^x \sqrt{1-e^2t}dt, \frac{e^x}{\sqrt{n-1}}[e^{i\sqrt{n-1}y_1}, ..., e^{i\sqrt{n-1}y_{n-1}}]].$$

It is straightforward to verify that $g(f_x, f_x) = 1$, $g(f_x, f_{y_i}) = 0$ and $g(f_{y_i}, f_{y_j}) = \delta_{ij}e^{2x}$, so it is an isometry. It is an immersion, and therefore a local embedding.

For the other two cases, it is sufficient to construct a local embedding of H^n into $Q_{\tilde{c}}^{2n-1}$, where either $-1 < \tilde{c} < 0$ or $\tilde{c} > 0$. A construction was outlined by W. Henke in [10] as follows: for H^n into S_R^{2n-1}, define

$$f(x, \mathbf{y}) = [\alpha(x)e^{i\beta(x)},\ \frac{e^x}{\lambda}e^{i\lambda y_1}, ...,\ \frac{e^x}{\lambda}e^{i\lambda y_{n-1}}]$$

The equations that we must satisfy are then

$$\begin{aligned} g(f, f) = \alpha^2(x) + \frac{(n-1)e^{2x}}{\lambda^2} &= R^2, \\ g(f_x, f_x) = (\alpha'(x))^2 + (\alpha(x)\beta'(x))^2 + \frac{4(n-1)e^{2x}}{\lambda^2} &= 1, \\ g(f_x, f_{y_i}) &= 0, \\ g(f_{y_i}, f_{y_j}) &= e^{2x}\delta_{ij}. \end{aligned}$$

The last two equations are automatically satisfied, and it is a simple matter to solve the first two (locally) for $\alpha(x)$ and $\beta(x)$, by taking the constant λ sufficiently large. To locally embed H^n into the hyperbolic space $H_{\tilde{c}}^{2n-1}$, one can use an almost identical construction, but here taking $H_{\tilde{c}}^{2n-1}$ as a subset of $\mathbf{R}^{2n}$ equipped with the Lorentzian inner product h, defined by $h(x, y) = -x_1y_1 + x_2y_2 + ... + x_{2n}y_{2n}$. It is well-known that the set $\{x \in (\mathbf{R}^{2n}, h)\ :\ h(x, x) = \tilde{c}^{-1}\}$, with the induced inner product (which becomes positive definite) is $H_{\tilde{c}}^{2n-1}$. Define f as above, except replacing $\alpha(x)e^{i\beta(x)}$ with $\alpha(x)[\cosh(\beta(x)), \sinh(\beta(x)]$. The construction then follows more or less as before.

Chapter 4

Complete Immersions of Negative Extrinsic Curvature

In contrast to the situation of positive extrinsic curvature, we will see that here local embeddability does not imply the existence of a complete immersion.

4.1 $\mathbf{0 = c < \tilde{c}}$: Euclidean into Spherical

This is the simple case: E^n can be completely immersed into a sphere S^{2n-1} of any radius r in the form of a Clifford torus $S^1_{r_1} \times ... \times S^1_{r_n}$, where $\sum r_i^2 = r^2$, with the metric induced from the sphere. Being a product of 1-dimensional manifolds, it is a flat embedding of the n-torus, which has E^n as it's universal cover.

4.2 $0 < \mathbf{c} < \tilde{\mathbf{c}}$: Spheres into Spheres

Here we have our first non-existence result, in spite of local existence. Namely, there is no complete isometric immersion of the sphere S^n_R into S^{2n-1}_r for $R > r$, since, by Moore's result, Theorem 3.1, the principal coordinates would define a diffeomorphism between R^n and the sphere S^n_R.

However, we *can* immerse any sphere S^n_R into S^{2n+1}_r, by composing a Clifford torus immersion of E^{n+1} into S^{2n+1}_r with the natural embedding of the sphere S^n_R in E^{n+1}.

This leave open the problem of an immersion of codimension n. See Lemma 5.1 below for a continuous embedding of S^n into S^{2n}_R, $R < 1$, which is a C^∞ isometry on the complement of a submanifold of dimension $n - 2$.

4.3 $\mathbf{c} < 0$: Immersing Hyperbolic Space - Some Constructions

This is where there are many questions still unanswered. It is known that complete immersions are always possible, but we don't appear to have a good idea what the smallest codimension is yet, except that it is not too large. The following proposition is an improvement by Wolfgang Henke on some results of Danilo Blanusa [2], [3].

Proposition 4.1 *(W. Henke [9], [10].)* *There is a complete isometric immersion of H^n into $Q_{\tilde{c}}^{4n-3}$ for all $\tilde{c} > -1$.*

Outline of the Proof: The proof is based on a modification of Blanusa's ([2], [3]) original embedding of H^2 into E^6, and of injective immersions (but not embeddings) of H^n into E^{6n-5} and into $S^{6n-4}_{\sqrt{\tilde{c}^{-1}}}$. In the chapter on embeddings below, we give a complete proof of an isometric embedding of H^n into E^{6n-6} which illustrates the idea. It would be helpful for the reader to read that before continuing here. Also compare with the local embeddings in Section 3.3.

On examination of the construction of the embedding $F: \ H^n \to E^{6n-6}$, it is clear that it is the simplest possible generalization of Blanusa's embedding $H^2 \to E^6$. Consequently, there is a lot of redundancy. First, by dropping the requirement that it be an embedding, one can construct an immersion of the form $(f_0(x), \mathbf{f}(x,y)): \ H^2 \to E^1 \times E^4 = E^5$. Using the upper half-plane model of H^2, with metric $\frac{1}{x^2}(dx^2 + dy^2)$, the function $\mathbf{f}$ which works is

$$\mathbf{f}(x,y) = \frac{1}{x}[\frac{\phi_1(x)}{s_1(x)}e^{is_1(x)y}, \ \frac{\phi_2(x)}{s_2(x)}e^{is_2(x)y}],$$

where $s_i(x)$ are piecewise constant and $\phi_i(x)$ are smoothing functions. By taking $s_i(x)$ sufficiently large on their intervals of constancy, it is not difficult to solve for $f_0(x)$ to make this an isometric immersion.

This extends naturally to an isometric immersion $H^n \to E^{4n-3}$

$$F(x, y_1, ..., y_{n-1}) = [f_0, \mathbf{f}(x, y_1), ..., \mathbf{f}(x, y_{n-1})].$$

For immersions into $Q_{\tilde{c}}^{4n-3}$, with $\tilde{c} \neq 0$, there is a similar map with an extra component, $H^n \to R^{4n-2}$

$$G(x, y_1, ..., y_{n-1}) = (f_0(x), f_1(x), \mathbf{f}(x, y_1), ..., \mathbf{f}(x, y_{n-1})),$$

but this time with an extra condition, namely $\tilde{g}(G, G) = \frac{1}{\tilde{c}}$, where $\tilde{g}$ is the Euclidean or Lorentzian inner product to get the spherical or hyperbolic case respectively. ♠

Remarks

- This gives us an isometric immersion of H^2 into E^5. It is still an open problem whether or not it can be done into E^4 (see below for E^3).
- It appears that for $n > 2$, all immersions have been simple generalizations of an isometric immersion $H^2 \to E^m$. It therefore seems likely that better results could be obtained for $n > 2$. A step in this direction, perhaps, is a result of Azov [1], which for $n > 2$ gives isometric immersions of H^n into E^{4n-4} and S^{4n-4}.

4.4 Non-Immersibility of the Hyperbolic Plane

To date we only have one non-existence result for complete immersions of H^n into $Q_{\tilde{c}}^{n+k}$, for $k \geq 2n-1$, and this only for H^2. But it is important, because it tells us that the local and global (immersion) problems are not the same.

Theorem 4.1 *(D. Hilbert [12]): There is no complete immersion of H^2 into $Q^3_{\tilde{c}}$, for any $\tilde{c} > -1$.*

Proof: Recall that by, Theorem 3.1, such an immersion would imply the existence of a diffeomorphism from $\mathbf{R}^2$ to H^2, given by the principal coordinates, whose direction fields are $X_i = \alpha_i e_i$. Using the umbilical inclusion of $Q^3_{\tilde{c}}$ into H^4, we have, by equations (3.5) and (3.6), for $i \neq j$:

$$\begin{aligned}
\bar{\nabla}_{e_i} e_j &= \sum_k \tau_j^k(e_i) e_k + \delta_{ij} \frac{\sqrt{\tilde{c}+1}}{\alpha_i} \xi_i, \\
\bar{\nabla}_{e_i} \xi_j &= -\delta_{ij} \frac{\sqrt{\tilde{c}+1}}{\alpha_i} e_i + \sum_k \omega_j^k(e_i) \xi_k, \\
\tau_i^j(e_j) &= \frac{1}{\alpha_j} e_i(\alpha_j), \\
\omega_i^j(e_j) &= \frac{1}{\alpha_j} e_j(\alpha_i),
\end{aligned}$$

where $\bar{\nabla}$ is the connection on H^4. The assumption that this is indeed the connection on H^4 means that we must have $g(\bar{R}(e_1, e_2)e_2, e_1) = -1$. Calculating this with the connection given above, one obtains the equation

$$\frac{1}{\alpha_1} e_2 e_2(\alpha_1) + \frac{1}{\alpha_2} e_1 e_1(\alpha_2) = 1 + \tilde{c}$$

Since $\alpha_1^2 + \alpha_2^2 = 1$, we may set $\alpha_1 = \cos\frac{\omega}{2}$, and $\alpha_2 = \sin\frac{\omega}{2}$, where $0 < \omega < \pi$, because α_i are both positive. We can rewrite the equation in terms of the coordinate directions and ω, to get

$$X_1 X_1(\omega) - X_2 X_2(\omega) = (1 + \tilde{c}) \sin\omega.$$

Changing to coordinates such that

$$\frac{\partial}{\partial u} = X_1 + X_2 = \alpha_1 e_1 + \alpha_2 e_2,$$
$$\frac{\partial}{\partial v} = X_1 - X_2 = \alpha_1 e_1 - \alpha_2 e_2,$$

the PDE is equivalent to

$$\frac{\partial^2 \omega}{\partial u \partial v} = (1+\tilde{c})\sin\omega, \qquad 0 < \omega < \pi. \tag{4.1}$$

Note that both $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ have unit length and that the angle between them is ω, therefore the area form on H^2 with respect to these coordinates is $dA = \sin\omega dudv$. So we can use the equation (4.1) to calculate the area of a rectangle in H^2 which is the image of $[a,b] \times [c,d]$ in these coordinates:

$$\begin{aligned} A &= \frac{1}{1+\tilde{c}} \int_a^b \int_c^d \frac{\partial^2 \omega}{\partial u \partial v} dudv \\ &= \frac{1}{1+\tilde{c}} \int_a^b (\frac{\partial \omega}{\partial u}(u,d) - \frac{\partial \omega}{\partial u}(u,c))du \\ &= \frac{1}{1+\tilde{c}} \{\omega(b,d) - \omega(a,d) - \omega(b,c) + \omega(a,c)\}. \end{aligned}$$

Since $0 < \omega < \pi$, it follows that the area of the rectangle is less than $\frac{2\pi}{1+\tilde{c}}$. But u and v are a global coordinate system for H^2, being a linear change of variables of the principle coordinates. This means that the area of the entire hyperbolic plane must be bounded by $\frac{2\pi}{1+\tilde{c}}$, and this is a contradiction, because it is easy to check that H^2 has infinite area. ♠

Remark: Hilbert's theorem was originally proved for the case $\tilde{c} = 0$. It has long been conjectured that an analogous result holds in higher dimensions, specifically that there is no complete immersion of H^n into E^{2n-1}. Some results have been obtained with special extra conditions - for example it has been proved that if a space-form M^n_{-1} is *not* simply connected then it cannot be immersed into E^{2n-1} [16]. However it is essential to the proof given that the fundamental group is non-trivial. See the survey [4] for other results.

Chapter 5

Global Embeddings of Negative Extrinsic Curvature

5.1 $0 < \mathbf{c} < \tilde{\mathbf{c}}$: Spheres inside Smaller Spheres

Any sphere of radius R and dimension n can be embedded in any sphere of smaller radius r of dimension $3n+2$ in the following way: define $f : E^{n+1} \to E^{3n+3}$ by

$$f(x_1, ..., x_{n+1}) = (ax_1, ..., ax_{n+1}, be^{ix_1}, ..., be^{ix_{n+1}})$$

where a and b are constants satisfying $a^2 + b^2 = 1$. Restricting f to $S^n_R \subset E^{n+1}$, we have

$$g(f, f) = a^2R^2 + (n+1)b^2,$$

$$g(f_{x_i}, f_{x_j}) = \delta_{ij}(a^2 + b^2) = \delta_{ij}.$$

Thus f is an isometry, and it is an embedding as long as we choose a non-zero. The image, $f(S_R^n)$ is contained in a sphere of radius $\sqrt{a^2R^2 + b^2(n+1)} = \sqrt{n+1}+\delta$, if we take a close to zero. Taking a sufficiently small, we can always embed an arbitrarily large sphere S_R^n into $S_{\delta+\sqrt{n+1}}^{3n+2}$, for δ arbitrarily close to zero, in this way. Rescaling, this proves the claim.

Remark: It should be possible to do much better than this. For example, if $n < 9$, and the difference in size of the spheres is not too large, the Clifford torus in S_r^{2n+1} contains an embedding of a sphere of radius $R > r$. Explicitly, define $f : E^{n+1} \to E^{2n+2}$ by

$$f(t_1, ..., t_{n+1}) = \frac{r}{\sqrt{n+1}}(e^{\frac{i\sqrt{n+1}}{r}t_1}, ..., e^{\frac{i\sqrt{n+1}}{r}t_{n+1}}).$$

As mentioned before, this is an isometric immersion, periodic in each t_i, with period $P = \frac{2\pi r}{\sqrt{n+1}}$, and an embedding when restricted to $D = (0, P) \times .... \times (0, P)$. We can fit an n-sphere of radius $R = P/2 - \epsilon = \frac{\pi r}{\sqrt{n+1}} - \epsilon$, where epsilon is arbitrarily small, into D, embedding S_R^n into S_r^{2n+1}. R will be larger than r if and only if $\pi > \sqrt{n+1}(1+\epsilon)$, that is, if and only if $n + 1 \leq 9$.

We also have:

Lemma 5.1 *There is a continuous embedding of S_r^n into S_R^{2n}, for any radius $R < r$ which is a C^∞ isometry at every point except on a submanifold of dimension $n-2$.*

Proof: It is sufficient to give an embedding of S^n into S_R^{2n} where R is any radius less than 1. An argument almost identical to the local argument (in a different codimension) given in Section 3.3.1 works: it is clear that the unit circle S^1 can be embedded isometrically into S^2_R for any R. Now assume that there is an embedding of the sort required, $f : S^{n-1} \to S^{2n-2}_r \subset E^{2n-1}$, where r is very small. Define $F : (0, \pi) \times S^{n-1} \to S^{2n}_R \subset E^{2n+1}$ by

$$F(x, \mathbf{y}) = [\alpha(x)e^{i\beta(x)}, \sin(x)f(\mathbf{y})].$$

Then, as in Section 3.3.1, F will be an isometry into S^{2n}_R if and only if the functions

$$\alpha(x) = \sqrt{R^2 - r^2\sin^2(x)},$$
$$\beta(x) = \int_0^x \frac{1}{\alpha(t)}\sqrt{1 - r^2(\cos^2 t + \frac{r^2\sin^2(2t)}{4\alpha^2(t)}})dt$$

are well defined, and this will be the case for small r. The only problem is that $U = (0, \pi) \times S^{n-1}$ is not quite the whole of S^n. The map $\phi : U \to S^n$ given by $\phi(t, \mathbf{x}) = (\sin(t)\mathbf{x}, \cos(t))$ is an isometry between U and $S^n - \{p, q\}$, where p, and q are the north and south poles respectively. We need to define F at at p and q, which is easy: the limits

$$\lim_{\mathbf{z}\to\mathbf{p}} F(z) = [R, 0, ..., 0],$$
$$\lim_{\mathbf{z}\to\mathbf{q}} F(z) = [Re^{i\beta(\pi)}, 0, ..., 0],$$

exist. Therefore defining F to have those values at p and q respectively gives a continuous function from the whole of S^n into S^{2n}_R, which is C^∞ except at the bad points of f, which are the product $(0, \pi) \times M$, and M is a submanifold of S^{n-1} of dimension $n-3$, and at the points p and q (one can check that the function is not differentiable at these points). ♠

Remark: Any immersion of S^2 into S^4 of the form $F:(0,\pi)\times S^1 \to S^4_R \subset E^5$,

$$F(x,\mathbf{y}) = [a(x), b(x), \sin(x) f(\mathbf{y})],$$

where f is an embedding of S^1 into S^2, will necessarily only be continuous at the North and South poles, because, in the isometry given above, $x = \cos^{-1}(y_3)$, where y_i are the coordinates of E^3, so $\sin(x) = \sqrt{1-y_3^2}$ and its first derivative blows up as $y_3 \to 1$.

5.2 $\mathbf{c \leq 0 < \tilde{c}}$: $\mathbf{H^n}$ and $\mathbf{E^n}$ into a Sphere

There can be no embedding of H^n or E^n into a sphere on topological grounds - the image would have to be compact, being a closed subset of a compact space. However, it is interesting to note that, as was previously mentioned, Blanusa constructed one-to-one immersions of H^n into spheres of arbitrary radius of dimension $(6n-5)$ [2], [3]. It is also possible to have a one-to-one immersion of E^n into a sphere, for example as a product $\gamma_1 \times ... \times \gamma_n \subset S^{4n-1}$, where each γ_i is a one-to-one immersion of the real line into a 2-torus.

5.3 Embedding $\mathbf{H^n}$ into $\mathbf{E^{6n-6}}$

We give here a complete proof of a generalization of Blanusa's isometric embedding of H^2 into E^6 [2]. The proof is representative of the methods used in the results of Blanusa and Henke.

Proposition 5.1 *(W. Henke, W. Nettekoven [11].) The metric* $g_{-1} = dx^2 + e^{2x}(dy_1^2 + \ldots + dy_{n-1}^2)$ *on* $\boldsymbol{R}^n$*, which is complete with constant sectional curvature* -1*. can be embedded into* E^{6n-6} *as the graph of a* C^∞ *function* $R^n \to R^{5n-6}$.

Proof: We will first give Blanusa's [2] embedding of H^2 into E^6. The metric $g_c = dx^2 + \cosh^2(x)dy^2$ is complete with constant curvature -1 on $\mathbf{R}^2$. This can be verified by checking that $\phi : (\mathbf{R}^2, g_c) \to (\mathbf{R}^2, g_{-1})$, defined by $\phi(x,y) = (-y + \ln(\cosh x), e^y \tanh x)$ is an isometry with inverse

$$\phi^{-1}(x,y) = (\sinh^{-1}(ye^x), \ln(\sqrt{e^{-2x} + y^2})).$$

We will embed $(\mathbf{R}^2, g_c)$ into E^6.

Lemma 5.2 *If* $\eta : \boldsymbol{R} \to \boldsymbol{R}$ *is any* C^∞ *function, then there exists a smooth function* $h : \boldsymbol{R}^2 \to E^4$ *such that the pull back of the metric* g_E *of* E^4 *is equal to*

$$h^* g_E = \epsilon(x)^2 dx^2 + \eta(x)^2 dy^2, \qquad 0 \le \epsilon(x) < \frac{1}{2}.$$

The proof of the lemma, which is really the clever part of the construction, relies on smoothed out step functions and will be given following the proof of the proposition.

Now apply the lemma to $\eta(x) = \sinh(x)$, and define $f: \mathbf{R}^2 \to E^6$ by

$$f(x,y) = [\int_0^x \sqrt{1-\epsilon(t)^2}dt, y, h(x,y)].$$

We compute:

$$g_E(f_x, f_x) = 1 - \epsilon^2(x) + g_E(h_x, h_x) = 1,$$

$$g_E(f_y, f_y) = 1 + g_E(h_y, h_y) = 1 + \sinh^2(x) = \cosh^2(x),$$

$$g_E(f_x, f_y) = g_E(h_x, h_y) = 0.$$

Thus $f^* g_E$ is the hyperbolic metric $dx^2 + \cosh^2(x)dy^2$. Moreover, since f_1 is a strictly increasing function of x alone, the projection onto the first two components of f is clearly a diffeomorphism. It follows that the image $f(\mathbf{R}^2)$ is the graph of a smooth function of two variables. In other words, f is a C^∞ isometric embedding of H^2 into E^6.

This embedding can be generalized to H^n into E^{6n-6} in a fairly simple way, however we need to use the metric g_{-1} given above, because if we try to generalize g_c to something like $dx^2 + \cosh^2(x)(dy_1^2 + ... + dy_{n-1}^2)$ then it no longer has constant sectional curvature. So first observe that the composition of f with ϕ^{-1}, namely

$$\tilde{f}(x,y) = [\int_0^{\sinh^{-1}(ye^x)} \sqrt{1-\epsilon(t)^2}dt, \ln(\sqrt{e^{-2x}+y^2}), h \circ \phi^{-1}(x,y)],$$

is, of course, an embedding $(\mathbf{R}^2, g_{-1}) \to E^6$. Using this, we define $F: (\mathbf{R}^n, g_{-1}) \to E^{6n-6}$ as follows:

$$f(x, \mathbf{y}) = \frac{1}{\sqrt{n-1}}(\tilde{f}(x, \sqrt{n-1}y_1), ...\tilde{f}(x, \sqrt{n-1}y_{n-1})].$$

The pull-back metric is again easy to calculate:

$$g_E(F_x, F_x) = (n-1)^{-1}[(n-1)(1-\epsilon^2(x)+\epsilon^2(x))] = 1,$$
$$g_E(F_{y_i}, F_{y_j}) = \delta_{ij}e^{2x},$$
$$g_E(F_x, F_{y_i}) = 0,$$

these values being inherited from the corresponding values for $\tilde{f}$. This shows that the pull-back metric is g_{-1}. Examination of the components F_1,F_2,F_7,F_{13},...,F_{5n-3} shows that one can recover x, y_1, ..., y_{n-1} from those n components. Consequently, the image of F is the graph of a function of n variables, and therefore an embedding. ♠

Proof of lemma (5.2): Define $h : \mathbf{R}^2 \to E^4$ by

$$h(x,y) = [\frac{\eta\psi_1}{s_1}(x)e^{iys_1(x)}, \frac{\eta\psi_2}{s_2}(x)e^{iys_2(x)}],$$

where $s_j(x)$ are piecewise constant functions, and $\psi_j(x)$ are C^∞ functions such that ψ_j vanishes, together with all of its derivatives, at each point of discontinuity of the corresponding s_j, and such that $\psi_1^2 + \psi_2^2 = 1$.

The function h is smooth and, bearing in mind that $s_i(x)$ are piecewise constant, we have:

$$h_x = [\frac{(\eta\psi_1)'e^{iys_1}}{s_1}, \frac{(\eta\psi_2)'e^{iys_2}}{s_2}]$$
$$h_y = [\eta\psi_1 ie^{iys_1}, \eta\psi_2 ie^{iys_2}].$$

The pull-back metric has coefficients:

$$
\begin{aligned}
g(h_x, h_x) &= (\frac{(\eta\psi_1)'}{s_1})^2 + (\frac{(\eta\psi_2)'}{s_2})^2 \;=:\; \epsilon(x), \\
g(h_y, h_y) &= \eta^2, \\
g(h_x, h_y) &= 0.
\end{aligned}
$$

The values of the step functions $s_j(x)$ on each interval of constancy are chosen independently from the values of $\psi_j(x)$, so we can choose s_j to be sufficiently large on each interval to make $\epsilon(x) < \frac{1}{2}$. ♠

Remarks:

• Formulae for the smoothing functions ψ_i are provided in Appendix A.

• This construction can, of course, be rescaled to give an embedding of H^n_c into E^{6n-6}, for any $c < 0$.

5.4 Embedding $\mathbf{H^n}$ into $\mathbf{H^{6n-5}_{\tilde{c}}}$

Any hyperbolic space H^n_c can be embedded into $H^{6n-5}_{\tilde{c}}$, where c and $\tilde{c}$ are arbitrary negative numbers, by composing an umbilic inclusion of E^{6n-6} into $H^{6n-5}_{\tilde{c}}$ with the embedding of H^n_c into E^{6n-6} described in the previous section.

Chapter 6

Conclusion

Curvatures	Local Embedding	Complete Immersion	Embedding
$c > \tilde{c}$	1	1	1
$c < \tilde{c} < 0$	$n-1$	$n-1 \leq k \leq 3n-3$	$n-1 \leq k \leq 5n-5$
$c < \tilde{c} = 0$	$n-1$	$n-1 \leq k \leq 3n-3$	$n-1 \leq k \leq 5n-6$
$c < 0 < \tilde{c}$	$n-1$	$n-1 \leq k \leq 3n-3$	Not Possible
$c = 0 < \tilde{c}$	$n-1$	$n-1$	Not Possible
$0 < c < \tilde{c}$	$n-1$	$n \leq k \leq n+1$	$n \leq k \leq 2n+2$

Table 6.1: Range of possible values for the smallest codimension k for which there exists an embedding of Q_c^n into $Q_{\tilde{c}}^{n+k}$.

The table above is a summary of the results we have proved. Note that the entries of the second column, rows two to four, should have n as a lower bound for the case $n = 2$, and that it is conjectured to be thus for all n. And for $n \geq 3$, the same column, rows three and four, have $3n-4$ as an upper bound, by [1]. Also recall that if we don't require an embedding to be a proper map, then the fourth and fifth row entries of the last column have affirmative, rather than negative, answers.

6.1 Open Problems

Ideally, one would like to replace all of the inequalities in Table 6 with single numbers. The hard part of this task, at least so far, seems to be getting the non-existence results. The equation we used to prove the non-immersibility of H^2 is actually the integrability condition for the corresponding connection. It's generalization to higher dimensions is a system of non-linear PDE's, the solutions of which would probably have to be better understood if one were to try a similar proof for general n.

However, there are still some very interesting questions which one could hope to solve in a shorter time frame. Here are some, the answers to which the author was unable to find in the literature:

1. Is there an isometric embedding or immersion of S^n into S^{2n}_r where r is less than 1?

2. In the cases where global embeddings are actually possible, is the minimum codimension needed for an embedding ever actually different from the codimension needed for a complete immersion? Given the results found so far, and

looking at the way they are constructed, one would expect this to be the case, however we have not found an instance where there is an immersion of Q_c^n into $Q_{\tilde{c}}^{n+k}$, and a proof that there is no embedding, except on topological grounds.

3. When c and $\tilde{c}$ are both non-zero, do their relative magnitudes make a difference? In Section 5.1 we were able to find, for $n < 9$, an embedding of a sphere S_R^n into S_r^{2n+1} with $R > r$, if the difference between R and r were not too great. However, we did not prove that this could not be done for arbitrary $R > r$ using some other method. Note that if this were the case, then the table of results might need to have considerably more rows, quite apart from the separate issue of whether there is a nice formula in terms of n for the codimensions required.

Appendix A

Smoothed Step Functions

The following construction is due to Blanusa [2]. Let $\chi(t) = \sin(\pi t)e^{-\sin^{-2}(\pi t)}$, (where we define $e^{-\sin^{-2}(0)}$ to have the value zero) and $A = \int_0^1 \chi(t)dt$. The functions

$$\psi_1(x) = \sqrt{\frac{1}{A}\int_0^{1+x} \chi(t)dt},$$
$$\psi_2(x) = \sqrt{\frac{1}{A}\int_0^{x} \chi(t)dt}$$

satisfy, for every $x \in \mathbf{R}$,:

$$\psi_1(x)^2 + \psi_2(x)^2 = 1. \tag{A.1}$$

Moreover, $\psi_i(x)$, together with all of its derivatives, vanishes at all integers congruent to i mod(2). Hence if $s_i(x)$ is a positive function which is piecewise constant, with discontinuities only at integer values congruent to i mod(2), then $\frac{\psi_i(x)}{s_i(x)}$ are smooth functions.

Proof: The functions $\psi_i(x)$ are even, so without loss of generality, assume $x > 0$.

In order to be sure that ψ_i are well-defined, we need to check that $\int_0^x \chi(t)dt$ is non-negative. By the periodicity of $\chi(t)$, it is enough to check this on the interval $[0,2]$. Non-negativity follows from the fact that χ is positive on $[0,1]$ and that, since $\chi(t+1) = -\chi(t)$, we have $|\int_1^x \chi(t)dt| \leq \int_0^1 (\chi(t)dt$ for $x \in [1,2]$.

Let us check (A.1). Observe that

$$A\psi_2^2(x) = \int_0^x \chi(t)dt = -\int_0^x \chi(t+1)dt = -\int_1^{1+x} \chi(t)dt.$$

Using this we obtain

$$\begin{aligned} A\psi_1^2(x) + A\psi_2^2(x) &= \int_0^{1+x} \chi(t)dt - \int_1^{1+x} \chi(t)dt \\ &= A, \end{aligned}$$

which proves (A.1).

Finally we must check that ψ_i and all its derivatives vanishes at $x = 2k+i$, for all integers k. Let us verify this for ψ_2 - then we will be done, because $\psi_1(x) = \psi_2(x+1)$. Clearly it is sufficient to check that the function $f(x) = \int_0^x \sin(t)e^{-\sin^{-2}(t)}dt$ has the desired properties at $x = 0$, taking all derivatives as limits from the right, and using that, by definition, $\psi_2(x) = \psi_2(-x)$ (note that ψ_i are periodic). Now $f(0) = 0$ and $f'(x) = \sin(x)e^{-\sin^{-2}(x)}$. It follows by induction that $f^{(n)}(x)$ is always the factor $e^{-\sin^{-2}(x)}$ divided by some rational function of $\sin(x)$ and $\cos(x)$. The numerator vanishes exponentially at $x = 0$, i.e. faster than any rational function of sines and cosines, which completes the proof.

Bibliography

[1] D. G. Azov. Isometric immersions of n-dimensional metrics into euclidean and spherical spaces. *Vestn. Chelyab. Univ., Ser. Mat. Mekh.*, 1 (2):12–16, 1994.

[2] D. Blanusa. Uber die einbettung hyperbolischer raume in euklidische raume. *Monatsh. Math.*, 59:217–229, 1955.

[3] D. Blanusa. Isometric imbedding of the hyperbolic n-space in a spherical (6n-4)-space. *Period. Math.-Phys. Astron. II. Ser.*, 19:53–61, 1964.

[4] A. A. Borisenko. Isometric immersions of space forms into riemannian and pseudo-riemannian spaces of constant curvature. *Russian Math. Surveys*, 56:3:425–497, 2001.

[5] E. Cartan. Sur les varietes de courbure constante d'un espace euclidean ou non euclidean. *Bull. Soc. Math. France*, 47, 48:125–160, 32–208, 1919, 1920.

[6] M. Dajczer and R. Tojeiro. Isometric immersions and the generalized laplace and elliptic sinh-gordon equations. *J. reine angew. Math.*, 467:109–147, 1995.

[7] M. P. Do Carmo. *Riemannian Geometry.* Birkhauser, 1992.

[8] M. L. Gromov and V. A. Rokhlin. Embeddings and immersions in riemannian geometry. *Russian Math. Surveys*, 25:1–58, 1970.

[9] W. Henke. Isometrische immersionen des n-dim. hyperbolischen raumes h^n in e^{4n-3}. *Manuscripta Math.*, 34:265–278, 1981.

[10] W. Henke. Isometric immersions of n-dim. hyperbolic spaces into (4n-3)-dim. standard spheres and hyperbolic spaces. *Math. Ann.*, 258:341–348, 1982.

[11] W. Henke and W. Nettekoven. The hyperbolic n-space as a graph in euclidean (6n-6)-space. *Manuscripta Math.*, 59:13–20, 1987.

[12] D. Hilbert. Ueber flachen von constanter gausscher krummung. *Trans. Amer. Math. Soc.*, 2:87–99, 1901.

[13] J. D. Moore. Isometric immersions of space forms in space forms. *Pacific J. Math.*, 40:157–166, 1972.

[14] J. Nash. C^1-isometric embeddings. *Ann. of Math. (2)*, 60:383–396, 1954.

[15] J. Nash. The embedding problem for riemannian manifolds. *Ann. of Math. (2)*, 63:20–63, 1956.

[16] Y. A. Nikolayevsky. A non-immersion theorem for a class of hyperbolic manifolds. *Differential Geom. Appl.*, 9:239–242, 1998.

[17] T. Otsuki. Isometric embedding of riemann manifolds in a riemann manifold. *J. Math. Soc, Japan*, 6:221–234, 1954.

[18] P. Petersen. *Riemannian Geometry*. Springer, 1998.